Spontaneous Symmetry Breaking in the SO(3) Gauge Theory to Discrete Subgroups

Gábor Etesi

Institute for Theoretical Physics
Eötvös Loránd University
Puskin u. 5-7, Budapest
H-1088 Hungary
etesi@hal9000.elte.hu

Abstract

In this paper we give a systematical description of the possible symmetry breakings in the $SO(3)$-gauge theory and show an algorithmical method to construct $SU(2)$- or $SO(3)$-invariant Higgs potentials in an arbitrary irreducible representation using regular graphs. We close our paper with the explicit construction of the Lagrangian of the simplest $SO(3) \to A_4$ theory.

PACS 11.15.-q
PACS 11.15.Ex

1 Introduction

A very interesting area of the today's theoretical physics is the study of the so-called discrete gauge theories (discrete Yang-Mills theories)[1],[2],[3]. A familiar way to construct such theories is to break down the continuous symmetry of a usual gauge theory using Higgs mechanism. But if we want to give an explicit example of such a symmetry breaking we need to solve two non trivial problems.

Firstly, how could we produce invariant polynomials of the initial gauge group in an arbitrary representation? This is a very hard problem of theory of group invariants and we cannot answer the question generally even in the very simple case of $SU(2)$.

Our second problem is to find a representation of the initial gauge group G for a given subgroup $H \subset G$ which possesses the symmetry breaking $G \to H$. Generally this is an algebro-geometrical question, because we can equivalently say that we must find points on the zero variety of the G-invariant polynomial having given stabilizer subgroup H under the action of G.

In the case of the group $SU(2)$ we were able to develop a simple method using regular graphs to make $SU(2)$-(and of course $SO(3)$-) invariant polynomials.

Because the subgroups of the group $SO(3)$ have contacts with regular two and three dimensional polyhedra, using simple methods from the theory of group representations we can list all possible stabilizer subgroups in an arbitrary irreducible representation of the group $SO(3)$.

2 Statement of the problem

If we want to break down the symmetry in a given gauge theory with gauge group G we need to give explicitly a so-called Higgs-potential V which is a polynomial in the Higgs scalar fields and satisfies the following conditions:

- V is invariant under the action of the group G in a given representation;
- V is bounded;
- the self interactions of the scalar fields induced by the polynomial V are renormalizable.

If we want to break the gauge symmetry to a given subgroup of the gauge group G we need some more information about the polynomial V. Our starting point is the familiar Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + (D^\mu\Phi)^\star(D_\mu\Phi) - V(\Phi). \tag{1}$$

Here $V : \mathbf{k}^n \to \mathbf{R}$ is a polynomial satisfies the above properties ($\mathbf{k}$ denotes $\mathbf{C}$ or $\mathbf{R}$). We can fix the minimum of the polynomial V to be zero. Let $Z(V = 0)$ denote the zero variety of V; so a vacuum state of the theory is given by $A_\mu^a = 0$ and $\Phi = \Phi_0 \in Z(V = 0)$. Using the potential V we can "break the symmetry spontaneously down" which means that the vacuum state (which is a point in $Z(V = 0)$) has no more the whole dynamical symmetry (the group G) but a subgroup H of G only. This subgroup H has the property that its elements stabilize the vacuum state (i. e. $H\Phi_0 = \Phi_0$). So this subgroup H is the *stabilizer subgroup* of the point $\Phi_0 \in Z(V = 0)$. We are interested in such situations when this group is *discrete*. Now we are in position to give a precise formulation of our problem.

Let us consider the field theory (1) with symmetry group G (We assume that this group is an algebraic subgroup of some $GL(\mathbf{k}^m)$) and let us take a discrete subgroup of it. Also take a representation $\rho : G \to GL(\mathbf{k}^n)$ of the group G. We are searching for polynomials $V : \mathbf{k}^n \to \mathbf{R}$ which satisfy the following conditions:

- V is invariant under the action ρ of G on $\mathbf{k}^n$;
- V is bounded;

- and $Z(V=0)$ has a subset with stabilizer subgroup $H \subset G$.

We have omitted the condition of renormalizability because this is a simple restriction of the degree of the polynomial V.

In the case $G = SO(3)$ we can solve the problem generally: we are able to list all possible symmetry violation and can show a simple algorithmical method to construct Higgs potentials in arbitrary high dimensional Higgs representations. Let us see how to do this!

3 Construction of invariant polynomials

Let j be an integer or half-integer number and let us take the space of all homogeneous complex polynomials $p_{2j}(x,y)$ having two variables and homogeneous degree $2j$. This space is naturally identified with $\mathbf{C}^{2j+1}$. By the aid of the canonical two dimensional representation of $SU(2)$ we can describe a $2j+1$ dimensional representations as follows. If $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in SU(2)$ the transformation of a vector $(x,y) \in \mathbf{C}^2$ given by

$$x \rightarrow \alpha x + \beta y,$$

$$y \rightarrow -\overline{\beta} x + \overline{\alpha} y.$$

Using this equations we get the transformation rule of a homogeneous polynomial:

$$\sum_{n=0}^{2j} a_n x^n y^{2j-n} \rightarrow \sum_{n=0}^{2j} a_n (\alpha x + \beta y)^n (-\overline{\beta} x + \overline{\alpha} y)^{2j-n}.$$

Clearly this is a $2j+1$ dimensional representation of the group $SU(2)$. The irreducibility of this representation is due to the fact that in the two dimensional representation of the $SU(2)$ there are not $SU(2)$-invariant polynomials. Let λ_n denote the roots of the polynomial $p_{2j}(x,1)$. Now we can write

$$\sum_{n=0}^{2j} a_n x^n y^{2j-n} = a_{2j} \prod_{n=1}^{2j} (x - \lambda_n y).$$

Definition. *Let k,l positive integer numbers The graph $\mathcal{G}$ is called k-**regular oriented graph of order** l if it has l vertices, in every vertex converge k edges and every edge has an orientation.*

For example **Figure 1** shows a 4-regular oriented graph of order 5.

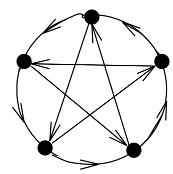

Figure 1.
4-regular graph of order 5.

Now let $\mathcal{G}$ be a k-regular oriented graph of order $2j$. We order to $\mathcal{G}$ an expression:

$$a_{2j}^k \prod_{(mn)} (\lambda_m - \lambda_n). \tag{2}$$

Here the product $\prod_{(mn)}$ is understood as the product (2) has to involve a factor $\lambda_m - \lambda_n$ if in its graph there is an edge of the form seen on **Figure 2**.

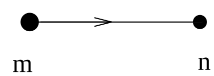

Figure 2.
A typical part of a graph.

(We can connect two vertices with more than one edges, in this situation we count the edges with multiplicity.)

On a graph of order $2j$ the permutation group S_{2j} acts naturally transposing the vertices of the graph. This gives the symmetrization of the expression (2).

Proposition 1. *The expression*

$$\frac{1}{|S_{2j}|} a_{2j}^k \sum_{\pi \in S_{2j}} \prod_{(mn)} (\lambda_{\pi(m)} - \lambda_{\pi(n)}) \tag{3}$$

is invariant under the action of $SU(2)$.$\diamondsuit$

Let σ_n denote the n^{th} elementary symmetric polynomial with variables $\lambda_1, ..., \lambda_{2j}$. The expression (3) clearly symmetric in $\lambda_1, ..., \lambda_{2j}$ so it is uniqly expressible as a polynomial in $\sigma_1, ..., \sigma_{2j}$. But using the well-known relations between the roots and coefficients of a polynomial, $\sigma_n = (-1)^n \frac{a_{2j-n}}{a_{2j}}$, we have the result that (3) is an $SU(2)$-(or, if j is an integer an $SO(3)$-) invariant polynomial of the form $f(a_0, ..., a_{2j})$ of homogeneous degree k.

The easy proof is left to the reader.

One can use this method very effectively if one has a computer (because of the symmetrization of the expression (2)). We have computed some invariants of $SU(2)$. Now we show only the 7 dimensional invariant of degree 2 illustrated on **Figure 3**.

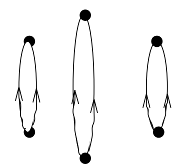

Figure 3.
$6a_3^2 - 16a_2a_4 + 40a_1a_5 - 240a_0a_6$
2-regular graph of order 6 and its $SU(2)$-invariant polynomial

4 Classification of symmetry violations of the $SO(3)$-theory

Now we turn to our second problem to classify all possible symmetry breaking in the $SO(3)$-gauge theory. Let $G \subset SO(3)$ a subgroup and its trivial representation given by $g \to 1 \in GL(1, \mathbf{R}) = \mathbf{R}^*$, $g \in G$. We say that $G \subset SO(3)$ is a maximal subgroup of the group $SO(3)$ if there is no subgroup $H \subset SO(3)$ satisfying $G \subset H$. The clue of the description is the following simple proposition.

Proposition 2. *Let $\rho : SO(3) \to GL(V)$ be an irreducible representation of the group $SO(3)$ and let G its maximal subgroup. If the direct decomposition of ρ according to G contains the trivial representation of G then in V there is a subspace W_G whose points are stabilized by the group G. Moreover the dimension of W_G is equal to the multiplicity of the trivial representation of G in ρ.*$\diamondsuit$

The straightforward verification of proposition 2 is left to the reader. So if the subgroup G satisfies the condition of proposition 2 the only thing what we need to do is to determine the characters of the group G in the representation ρ. But this is not difficult. Firstly if we take a review about the (discrete) subgroups of $SO(3)$ we find that these groups are closely related to well-known geometrical objects: these groups are the symmetry groups of the two and three dimensional regular polyhedra. If we take into account this fact we are able to construct these groups as a set of rotations under which the adequate regular polyhedron remains invariant (but not pointwise).

But if we know these rotations we can easily give the characters of the subgroup G in the representation ρ since

$$\chi_j(\phi) = \frac{\sin(j+\frac{1}{2})\phi}{\sin\frac{\phi}{2}}.$$

Here j denotes the weight of the representation ρ.

If the group is not maximal i.e. exists a subgroup H such that $G \subset H \subset SO(3)$, then we need to consider the multiplicity of the trivial representation of G respectively of H. If the multiplicity of the trivial G-representation is bigger than the multiplicity of the trivial H-representation then there are points in V whose stabilizer subgroup is the not-maximal subgroup G. Leaving some technical details we get in summary the table at the end of the paper.

5 The $SO(3) \to A_4$ theory

The time has come to examine explicitly a not usual symmetry violation. Using the Table at the end of the paper we can see: it is possible to violate the $SO(3)$ gauge symmetry to its non-Abelian subgroup A_4 using 7 dimensional Higgs representation. We choose for this procedure the potential showed on **Figure 3**. The 7 dimensional representation is constructed by the above polynomial method and is the 7 dimensional complex irreducible representation of the group $SO(3)$, too. Firstly we need to find a 7 dimensional *real* irreducible subspace $\mathbf{R}^7$ of $\mathbf{C}^7$ which gives the *real* representation of the $SO(3)$. Not difficult to see that a simple basis of this real subspace is given by the polynomials which satisfy the functional equation

$$p(x,y) = -\overline{p}(-\overline{y}, \overline{x}).$$

If we want to get an orthogonal real representation we need to multiply these vectors by certain numerical factors and get:

$$\begin{gathered}
\frac{i}{\sqrt{120}}(x^6+y^6); \quad \frac{1}{\sqrt{120}}(x^6-y^6); \\
\frac{1}{\sqrt{20}}(x^5y+xy^5); \quad \frac{i}{\sqrt{20}}(x^5y-xy^5); \\
\frac{i}{\sqrt{8}}(x^4y^2+x^2y^4); \quad \frac{1}{\sqrt{8}}(x^4y^2-x^2y^4); \\
\frac{1}{\sqrt{3}}x^3y^3.
\end{gathered} \tag{5}$$

In this basis the polynomial illustrated on **Figure 3** has the simple form

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2$$

which shows that this polynomial is bounded. Now we turn to our next question: how to find the coordinates of the A_4 vacuum? The group A_4, the 4^{th} alternating group, has two generators denoted by a, b. Clearly, our points need to be in

the linear space $\mathbf{R}^7 \cap \mathbf{Ker}(\rho(a) - Id) \cap \mathbf{Ker}(\rho(b) - Id)$ and have to be normed. After constructing the 7 dimensional representation of a and b we get the two possible vacuum states in the basis (5):

$$\pm \frac{1}{\sqrt{270}} \begin{pmatrix} \sqrt{120} \\ 0 \\ 0 \\ 5\sqrt{6} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (6)$$

Now we can write up the Lagrangian of the simplest $SO(3) \rightarrow A_4$ theory:

$$\mathcal{L} = -\frac{1}{4} F_i^{\mu\nu} F^i_{\mu\nu} + (D^\mu \Phi)^\star (D_\mu \Phi) - \lambda(\Phi_0^2 + ... + \Phi_6^2 - 1)^2. \qquad (7)$$

Using (6) and (7) together we are able to study this "exotic" non-Abelian discrete gauge theory.

6 Conclusions

In our paper we have studied the $SO(3)$ gauge theory. We have developed a general method to construct $SO(3)$-invariant polynomials and have given a systematical description of the possible symmetry violations in the $SO(3)$ theory. Our results are important because it is possible that a general discrete gauge theory in two spacetime dimensions possesses a strange field theoretical symmetry, the so-called quantum symmetry [2],[3]. By the aid of explicit examples like the $SO(3) \rightarrow A_4$ model we can study this question very effectively.

7 References

[1] M.G. Alford, J. March-Russel, F.Wilczek: *Discrete quantum hair on black holes and the non-Abelian Aharonov-Bohm effect*,in: Nucl. Phys. B **B 337**, no. 3 (1990), 695-708.

[2] F. A. Bais, P. van Driel, M.de Wild Propitius: *Anyons in discrete gauge theories with Chern-Simons terms*, in: Nucl. Phys., **B893**, (1993), 547.

[3] F. A. Bais, P. van Driel, M.de Wild Propitius: *Quantum symmetries in discrete gauge theories*, in: Phys. Lett., **B280**, (1992), 63-70.

The following table shows the stabilizer subgroups of the group $SO(3)$ in an arbitrary $2j+1$ dimensional irreducible representation.

The representation of weight j contains the following stabilizer subgroups systematically:

- if j is even and $j \geq 4$, then:

$$\mathbf{1}, \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_3, ..., \mathbf{Z}_j, D_3, ...D_j, O(2), SO(3),$$

- if j is odd and $j \geq 5$, then:

$$\mathbf{1}, \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_3, ..., \mathbf{Z}_j, D_3, ..., D_j, SO(2), SO(3),$$

Beyond these the not systematical groups are showed on **Table 1**.

The higher dimensional representations one can simple list:

- if $j \geq 30$ and is even the stabilizer subgroups in the $2j+1$ dimension al representations are:

$$\mathbf{1}, \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_3, ..., \mathbf{Z}_j, D_3, ..., D_j, A_4, A_5, S_4, O(2), SO(3),$$

- if $j \geq 31$ and is odd then we get:

$$\mathbf{1}, \mathbf{Z}_2 \oplus \mathbf{Z}_2, \mathbf{Z}_3, ..., \mathbf{Z}_j, D_3, ..., D_j, A_4, A_5, S_4, SO(2), SO(3).$$

$\dim\rho$	1	3	5	7	9	11	13	15	17	19
H	$SO(3)$	$SO(2)$ $SO(3)$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$ $O(2)$ $SO(3)$	$\mathbf{1}$ A_4 $\mathbf{Z}_3$ D_3 $SO(2)$ $SO(3)$	S_4		A_4 S_4 A_5	A_4	S_4	A_4 S_4

$\dim\rho$	21	23	25	27	29	31	33	35	37	39
H	A_4 S_4 A_5	A_4	A_4 S_4	A_4 S_4	A_4 S_4	A_4 S_4 A_5	A_4 S_4 A_5	A_4 S_4	A_4 S_4 A_5	A_4 S_4

$\dim\rho$	41	43	45	47	49	51	53	55	57	59
H	A_4 S_4 A_5	A_4 S_4 A_5	A_4 S_4 A_5	A_4 S_4	A_4 S_4 A_5	A_4 S_4 A_5	A_4 S_4 A_5	A_4 S_4 A_5	A_4 S_4 A_5	A_4 S_4

Table 1.

The not-systematical stabilizer subgroups in the low dimensional irreducible representations of the group $SO(3)$